WESLEY COLLEGE

By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST
SEMESTER ONE 2018
TEST 2: SOLUTIONS Functions and Graphs

Name: ______________________

Monday 26 March

Time: 55 minutes **Mark** **/ 50 =** %

- Answer all questions neatly in the spaces provided. **Show all working.**
- You are permitted to use the Formula Sheet in **both** sections of the test.
- You are permitted one A4 page (one side) of notes in the calculator assumed section.

Calculator free section **Suggested time: ~25minutes** **/25**

Question 1 (6 marks)

The function f is given by $f(x) = \ln(3x - 6), x \in \mathbb{R}, x > 2$

a) Find $f^{-1}(x)$.

$y = \ln(3x-6)$

$x = \ln(3y-6)$

$e^x = 3y - 6$

$\Rightarrow y = \frac{1}{3}(e^x + 6)$ $\qquad f^{-1}(x) = \frac{1}{3}(e^x+6)$

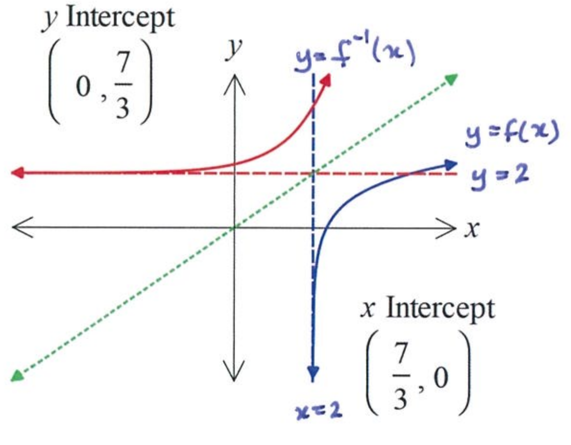
[2]

b) State the domain and range of $f^{-1}(x)$.

$x \in \mathbb{R}$ $\qquad y > 2$

[2]

c) Sketch the graphs of $f(x)$ and $f^{-1}(x)$, noting where the graphs intersect the axes. Clearly mark any asymptotes.



[2]

Question 2 (10 marks)

$$h(x) = 4 - x^2, \quad k(x) = \sqrt{1 - x^2} \quad \text{and} \quad l(x) = \frac{1}{x}.$$

a) Evaluate $h \circ l\left(\frac{1}{2}\right)$

$h\left(\frac{1}{1/2}\right) = h(2) = 4 - 2^2 = 0$

[1]

b) State the domain of $y = k(x)$

$-1 \leq x \leq 1$

[1]

c) Determine the domain and range of

i. $h \circ k(x)$

$h \circ k(x) = 4 - \left(\sqrt{1-x^2}\right)^2$
$= 3 + x^2$

Domain $-1 \leq x \leq 1$

Range $3 \leq h \circ k(x) \leq 4$

$h(x)$	$k(x)$
$x \in \mathbb{R}$	$-1 \leq x \leq 1$
$y \leq 4$	$0 \leq y \leq 1$

[2]

ii. $l \circ h(x)$

$l \circ h(x) = \frac{1}{4-x^2}$

Domain $x \neq \pm 2$

Range $y < 0 \quad y \geq \frac{1}{4}$

$h(x)$	$l(x)$
$x \in \mathbb{R}$	$x \neq 0$
$y \leq 4$	$y \neq 0$

[3]

d) Does $h(x)$ have an inverse? Justify your reasoning, mathematically.

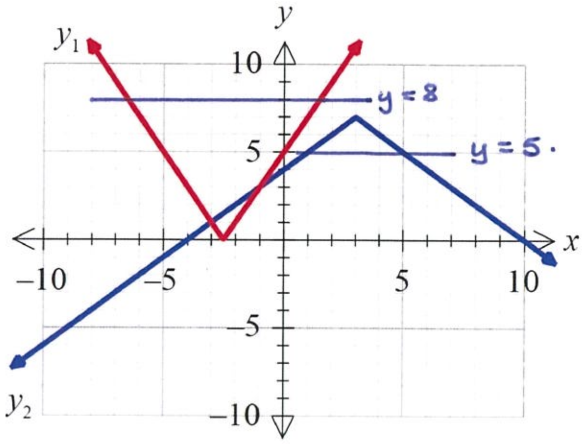
$h(x)$ does not have an inverse

It is a many-to-one function & does not pass the horizontal line test.
(For it to have an inverse, the natural domain of $h(x)$ would have to be restricted.)

[3]

Question 3 (5 marks)

The graphs of y_1 and y_2 are shown in the diagram.



a) Use the graphs to solve the following equations.

i. $y_1 = 8$ $\quad x = \frac{-13}{2}, \quad x = \frac{3}{2}.$

ii. $y_2 \leq 5$ $\quad x \leq 1$ or $x \geq 5$

iii. $y_1 = y_2$ $\quad x = -3, \quad x = -1.$

[3]

b) State the equation for the graph of:

i. $y_1 = |2x + 5|$

ii. $y_2 = 7 - |3 - x|.$

[2]

Question 4 (4 marks)

Given the functions $f(x) = (2x - 1)(x + 3)$ and $g(x) = 6x^2 + 19x - 36$, determine the following, justifying your answers:

a) the domain of $h(x)$ where $h(x) = \frac{f(x)}{g(x)}$

Consider the denominator in factored form

$6x^2 + 19x - 36$

$= (3x - 4)(2x + 9) \qquad \therefore \quad x \in \mathbb{R} : x \neq \frac{4}{3} \quad x \neq -\frac{9}{2}$

[2]

b) the behaviour of $h(x)$ as $x \to \pm\infty$

Consider the dominant powers.

$h(x) = \dfrac{2x^2 + \ldots}{6x^2 + \ldots} \qquad \therefore \quad h(x) \to \frac{1}{3}.$

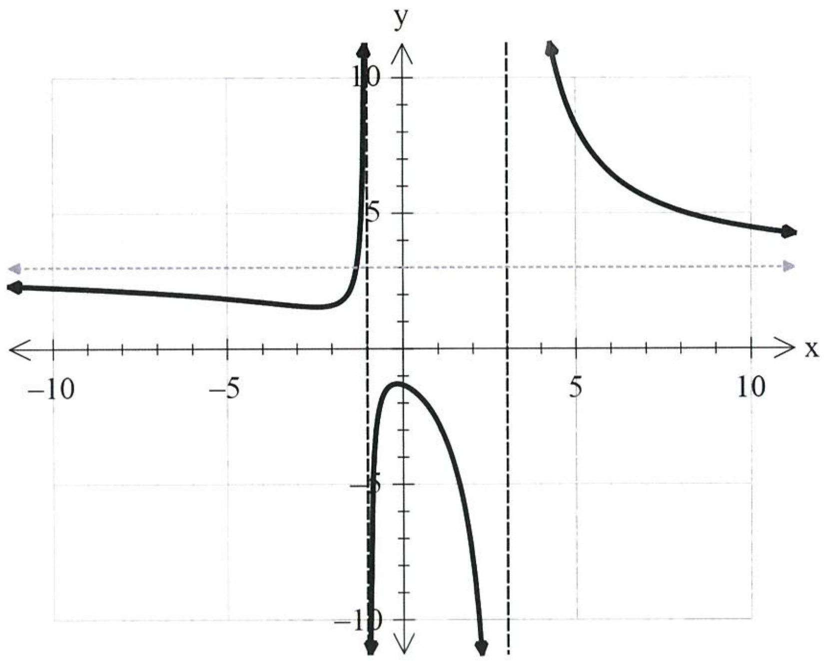
[2]

Name: ____________________

Calculator assumed section **Suggested time: ~25 minutes** /25

Question 5 (5 marks)

The graph of the rational function $f(x) = \frac{ax^2+bx+c}{x^2+dx+e}$ is drawn below.



Given $f\left(\frac{1}{2}\right) = \frac{-9}{5}$ and the y-intercept is $\frac{-4}{3}$,

find, with reasoning, the values of a, b, c, d and e.

vertical asymptotes @ $x = -1$ $x = 3$ $(x+1)(x-3) = x^2 - 2x - 3$

$d = -2$

$e = -3$

Horizontal asymptote @ $y = 3$ $\frac{ax^2+\dots}{x^2+\dots}$ $\Rightarrow a = 3$

y-intercept at $(0, -\frac{4}{3})$ $-\frac{4}{3} = \frac{c}{e} = \frac{c}{-3}$ $\Rightarrow c = 4$

substitute $(\frac{1}{2}, -\frac{9}{5})$ into $f(x)$

$f(\frac{1}{2}) = \frac{3x^2 + bx + 4}{x^2 - 2x - 3} = -\frac{9}{5}$ $\Rightarrow b = 4.$

[5]

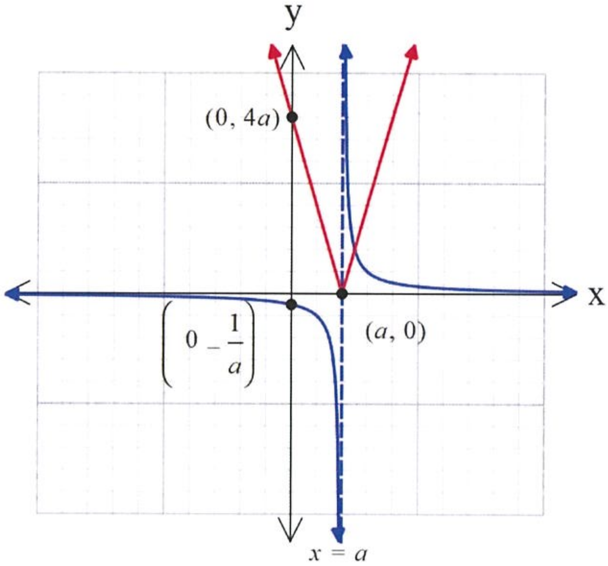
Question 6 (7 marks)

a) On the same diagram sketch the graphs of

$$y = \frac{1}{x-a} \text{ and } y = 4|x-a|$$

where a is a positive constant.

Show clearly the coordinates of any points of intersection with the axes.



✓✓ for graphs

$(a,0)$ $(0,4a)$ ✓

$x=a$ $(0,-\frac{1}{a})$ ✓

[4]

b) Hence, or otherwise, find the set of values of x for which $\frac{1}{x-a} < 4|x-a|$.

From graph: observe when $x < a$ ✓

Point of intersection:

$$\frac{1}{x-a} = 4(x-a)$$

$$\Rightarrow \frac{1}{4} = (x-a)^2$$

$$x-a = \frac{1}{2}$$

$$x = a + \frac{1}{2}$$

$x > a + \frac{1}{2}$. ✓✓

[3]

Question 7 (5 marks)

Given $g(x) = ax + b, a > 0,\quad g^2(4) = 12,\quad g^{-1}(3) = 3$, determine the values of a and b.

$\Rightarrow g(3) = 3$

$g^2(x) = a(ax+b) + b$
$= a^2x + ab + b.$ ✓

$\therefore 3 = 3a + b$ ✓

$g^2(4) = 4a^2 + ab + b = 12$ ✓

Solve $\left.\begin{array}{l} 4a^2 + ab + b = 12 \\ 3 = 3a + b \end{array}\right\}$

$a = 3 \quad (a > 0)$ ✓✓
$b = -6.$

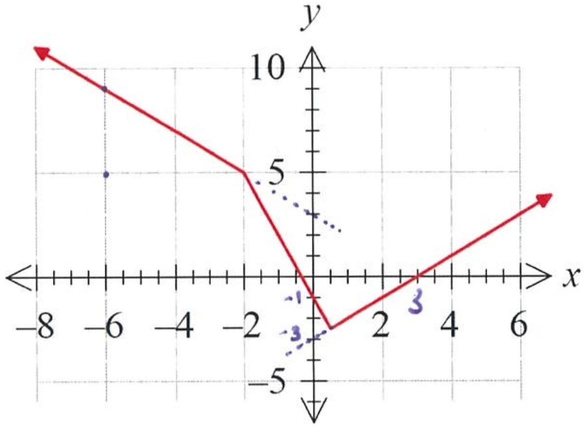
[5]

Question 8 (5 marks)

$f(x) = |2x - 1|$ and $g(x) = |x + 2|$

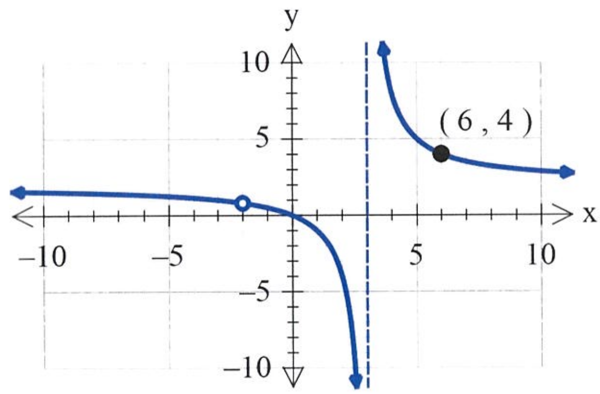
Determine a piecewise defined expression for $h(x) = f(x) - g(x)$ and sketch $h(x)$ on these axes.

$$h(x) = \begin{cases} 3 - x & x < -2 \\ -3x - 1 & -2 \leq x < 0.5 \\ x - 3. & x \geq 0.5 \end{cases}$$



[5]

Question 9 (3 marks)



This graph represents a function of the form

$$f(x) = \frac{kx(x+a)}{(x+a)(x+b)}$$

Determine the values of a, b and k referring to properties of the graph to justify your reasoning.

Point discontinuity at $x = -2 \quad \Rightarrow \quad a = 2$

Vertical asymptote at $x = 3 \quad \Rightarrow \quad b = -3$

using either $(6,4)$ or horizontal asymptote $y = 2$ (behaviour as $x \to \pm\infty$) $\quad k = 2.$

[3]